Int. J. Solids Structures Vol. 28, No. 2, pp. 161–177, 1991
Printed in Great Britain.

0020-7683/91 $3.00+.00

TWO-DIMENSIONAL THEORIES DEDUCED FROM THREE-DIMENSIONAL THEORY FOR A TRANSVERSELY ISOTROPIC BODY—II. PLANE PROBLEMS

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(*Received* 8 *February* 1990; *in revised form* 28 *July* 1990)

Abstract—Various two-dimensional equations for plane problems have been deduced systematically and directly from the three-dimensional theory of transversely isotropic bodies without any *ad hoc* assumptions. These equations can be used to construct new refined theories for the plane problems. In the case of homogeneous boundary conditions, the equations obtained are exact in the sense that a solution of them will satisfy all the balance equations of the three-dimensional theory. In the case of nonhomogeneous boundary conditions, the approximate equations are accurate up to the second-order terms with respect to plane thickness. The results of this paper also verify the stress assumption in the classical plane stress problem.

1. INTRODUCTION

This paper presents the second part of my research on deducing two-dimensional theories from three-dimensional theory for a transversely isotropic body. We have shown in the first part (Wang, 1989a) that the general deformation of a transversely isotropic body can be decomposed into two parts: the anti-symmetric deformation and the symmetric deformation. The two-dimensional problems associated with the two types of deformation have been called plate and plane problems, respectively. With the first part of this research focusing on the plate problems, we will investigate the plane problems in this paper.

The method used to deduce the two-dimensional theories from three-dimensional theory directly was originally introduced by Cheng (1979) for the development of refined plate theories. The new results obtained by Wang (1985, 1989a,b,c) and by Barrett and Ellis (1988) have indicated that application of Cheng's method in plate theory is quite successful. A more general procedure of deriving approximate lower-dimensional theories from a higher-dimensional theory with the aid of symbolic computation has been proposed by Wang (1989e).

Unlike the plate problems, the equations of classical two-dimensional plane problems for isotropic materials are well developed in theory and widely used for various engineering problems. The successful application of the classical plane theory makes it seem unnecessary to establish any refined theories, as have been proposed for classical plate theory. Consequently, any attempt to suggest a new plane theory may result in some controversial (or, hopefully, significant) issues and effort toward this has hardly been found in the literature. As the first step toward refined plane theories, we will concentrate only on two topics in this paper: (i) to derive new and exact two-dimensional equations directly from the three-dimensional theory for the classical plane problems; and (ii) to generalize the classical plane problems by allowing nonhomogeneous boundary conditions at the top and bottom surfaces of planes and then give the corresponding two-dimensional approximate equations. It is not our purpose to develop a complete refined theory for plane problems here. Some important issues for a complete refined theory, such as the specification of boundary conditions at edges, are open for future research.

Section 2 gives the exact two-dimensional plane equations of infinite order for transversely isotropic bodies. These equations will serve as the basis for the derivation of the finite order two-dimensional plane equations later. Sections 3 and 4 present the major results of this paper: the exact plane equations for homogeneous boundary conditions and the approximate plane equations for nonhomogeneous boundary conditions. Finally, we summarize the paper and discuss some issues for future research in Section 5.

2. EXACT TWO-DIMENSIONAL PLANE EQUATIONS FOR A TRANSVERSELY ISOTROPIC BODY

Let us consider a linear and transversely isotropic elastic body occupying a domain of $\Omega \times \{h/2 \geqslant z \geqslant -h/2\}$ in a rectangular coordinate system (x, y, z), where the x–y plane is the plane of isotropy and Ω is an arbitrary region on it. Let U, V and W be three-dimensional displacements of the transversely isotropic elastic body in the x, y and z directions, respectively. Since this paper is to deal with the plane problems, we will only consider the symmetric deformation of the transversely isotropic body caused by a set of symmetric surface loads with respect to the x–y plane.

According to Wang (1989a), the three-dimensional displacements U, V and W in this case can be represented in terms of the mid-plane displacements u, v and rotation ψ as

$$U = CS_0 u + \partial_x L_u(\partial_x u + \partial_y v) - \partial_x L_{1\psi}\psi, \quad V = CS_0 v + \partial_y L_u(\partial_x u + \partial_y v) - \partial_y L_{1\psi}\psi,$$
$$W = L_\psi \psi - L_{2u}(\partial_x u + \partial_y v), \tag{1}$$

where

$$u = U|_{z=0}, \quad v = V|_{z=0}, \quad \psi = \partial_z W|_{z=0},$$

and the differential operators are defined to be

$$\partial_x(\ ) = \frac{\partial(\ )}{\partial x}, \quad \partial_y(\ ) = \frac{\partial(\ )}{\partial y}, \quad \partial_z(\ ) = \frac{\partial(\ )}{\partial z}, \quad L_u = s_0^2 CC_0 - \frac{\beta}{\alpha}\Omega^1_{cc} + \frac{\beta}{\gamma}\Omega^0_{cc},$$

$$L_\psi = z - \nabla^2\left(\Omega^2_{ss} - \frac{\beta}{\alpha}\Omega^1_{ss}\right), \quad L_{1\psi} = \frac{1}{\alpha}\Omega^1_{cc}, \quad L_{2u} = \frac{\beta}{\alpha\gamma}\nabla^2[\alpha\gamma\Omega^2_{ss} - (\alpha^2 + \beta\gamma)\Omega^1_{ss} + \alpha\beta\Omega^0_{ss}].$$

The coefficients α, β, γ and the differential operators CS_i, SN_i, SS_i, CC_i, Ω^i_{ss}, Ω^i_{cc} are given in Appendix A. Note that the differential operators must be interpreted as representing series in powers of operators $(s_i\nabla)^2$.

When a transversely isotropic body reduces to an isotropic body, the differential operators L_u, L_{2u}, L_ψ and $L_{1\psi}$ in (1) become

$$L_u = -\frac{z}{2\alpha}\frac{\sin\nabla z}{\nabla}, \quad L_{2u} = \frac{1}{2\alpha}\left(z\cos\nabla z - \frac{\sin\nabla z}{\nabla}\right), \quad L_\psi = \frac{\sin\nabla z}{\nabla} - L_{2u}, \quad L_{1\psi} = -L_u;$$

and expression (1) reduces to the form

$$\begin{pmatrix} U \\ V \end{pmatrix} = \cos\nabla z \begin{pmatrix} u \\ v \end{pmatrix} - \frac{1}{2\alpha}\frac{z\sin\nabla z}{\nabla}\begin{pmatrix} \partial_x e \\ \partial_y e \end{pmatrix}$$
$$W = \frac{\sin\nabla z}{\nabla}\psi - \frac{1}{2\alpha}\left(z\cos\nabla z - \frac{\sin\nabla z}{\nabla}\right)e, \tag{2}$$

where $e = \partial_x u + \partial_y v + \psi$, $\alpha = 1 - 2v$ and v is Poisson's ratio.

The corresponding stresses follow from constitutive equations (see Appendix A):

$$\gamma A_{13}^{-1}\sigma_{xx} = [\mu\beta(CS_0 + \nabla^2 L_u) - 2\alpha\mu s_0^2\partial_{yy}L_u - \gamma\partial_z L_{2u}](\partial_x u + \partial_y v) - 2\alpha\mu s_0^2 CS_0\partial_y v$$
$$- (\mu\beta\nabla^2 L_{1\psi} - \gamma\partial_z L_\psi - 2\alpha\mu s_0^2\partial_{yy}L_{1\psi})\psi, \tag{3}$$

$$\gamma A_{13}^{-1}\sigma_{yy} = [\mu\beta(CS_0+\nabla^2 L_u)-2\alpha\mu s_0^2\partial_{xx}L_u-\gamma\partial_z L_{2u}](\partial_x u+\partial_y v)-2\alpha\mu s_0^2 CS_0\partial_x u$$
$$-(\mu\beta\nabla^2 L_{1\psi}-\gamma\partial_z L_\psi-2\alpha\mu s_0^2\partial_{xx}L_{1\psi})\psi, \quad (4)$$

$$A_{13}^{-1}\sigma_{zz} = (CS_0+\nabla^2 L_u-\mu\partial_z L_{2u})(\partial_x u+\partial_y v)-(\nabla^2 L_{1\psi}-\mu\partial_z L_\psi)\psi, \quad (5)$$

$$A_{66}^{-1}\sigma_{xy} = CS_0(\partial_x v+\partial_y u)+\partial_{xy}L_u(\partial_x u+\partial_y v)-\partial_{xy}L_{1\psi}\psi, \quad (6)$$

$$A_{44}^{-1}\sigma_{xz} = -s_0^2\nabla^2 SN_0 u+(\partial_z L_u-L_{2u})\partial_x(\partial_x u+\partial_y v)-(\partial_z L_{1\psi}-L_\psi)\partial_x\psi, \quad (7)$$

$$A_{44}^{-1}\sigma_{yz} = -s_0^2\nabla^2 SN_0 v+(\partial_z L_u-L_{2u})\partial_y(\partial_x u+\partial_y v)-(\partial_z L_{1\psi}-L_\psi)\partial_y\psi. \quad (8)$$

The symmetric surface loading conditions on the two surfaces $z = -h/2$ and $h/2$ can be specified as:

$$\sigma_{zz}(x,y,h/2) = p_s(x,y), \quad \sigma_{zz}(x,y,-h/2) = p_s(x,y),$$
$$\sigma_{xz}(x,y,h/2) = q_{xs}(x,y), \quad \sigma_{xz}(x,y,-h/2) = -q_{xs}(x,y),$$
$$\sigma_{yz}(x,y,h/2) = q_{ys}(x,y), \quad \sigma_{yz}(x,y,-h/2) = -q_{ys}(x,y); \quad (9)$$

which lead to the following system of linear differential equations for the mid-plane displacements and rotation (u, v, ψ):

$$\begin{bmatrix} \Sigma_{11}\partial_x & \Sigma_{11}\partial_y & -\Sigma_{13} \\ -s_0^2\nabla^2 SN_0+\Sigma_{22}\partial_{xx} & \Sigma_{22}\partial_{xy} & -\Sigma_{33}\partial_x \\ \Sigma_{22}\partial_{xy} & -s_0^2\nabla^2 SN_0+\Sigma_{22}\partial_{yy} & -\Sigma_{33}\partial_y \end{bmatrix}\begin{pmatrix} u \\ v \\ \psi \end{pmatrix} = \begin{pmatrix} A_{13}^{-1}p_s \\ A_{44}^{-1}q_{xs} \\ A_{44}^{-1}q_{ys} \end{pmatrix}, \quad z = h/2, \quad (10)$$

where

$$\Sigma_{11} = CS_0+\nabla^2 L_u-\mu\partial_z L_{2u}, \quad \Sigma_{13} = \nabla^2 L_{1\psi}-\mu\partial_z L_\psi, \quad \Sigma_{22} = \partial_z L_u-L_{2u},$$
$$\Sigma_{33} = \partial_z L_{1\psi}-L_\psi, \quad z = h/2. \quad (11)$$

Equations (10) represent the exact two-dimensional equations of the plane problems for a transversely isotropic elastic body. Obviously, when applying the homogeneous boundary conditions, we obtain the exact equations for the plane stress problem in classical two-dimensional elasticity (Timoshenko and Goodier, 1969). To consider the plane strain problem, we have to use the symmetric surface displacement or strain conditions (Wang, 1989d).

However, in general, these equations are not applicable since they are of infinite order. To develop a set of practical governing equations of finite order for the plane problems, we will study the general solution of (10) first.

Let D_P be the determinant and D_{ij} (given in Appendix A) be the cofactors of the differential operator matrix in (10). The general solution of (10) can be expressed as (Wang 1989a)

$$u = \sum_{i=1}^{3} D_{i1}\Phi_i, \quad v = \sum_{i=1}^{3} D_{i2}\Phi_i, \quad \psi = \sum_{i=1}^{3} D_{i3}\Phi_i; \quad (12)$$

the values of Φ_i satisfying the differential equations

$$D_P\Phi_i = X_i \quad (i = 1, 2, 3) \quad (13)$$

where

$$X_1 = A_{13}^{-1}p_x, \quad X_2 = A_{44}^{-1}q_{xs}, \quad X_3 = A_{44}^{-1}q_{ys}.$$

After some tedious algebraic manipulation, the determinant D_P is found to be

$$D_P = s_0^2\nabla^4 SN_0G_0, \quad z = h/2, \tag{14}$$

where

$$G_0 = \Sigma_{13}\Sigma_{22} - \Sigma_{11}\Sigma_{33} - s_0^2SN_0\Sigma_{13}. \tag{15}$$

For an isotropic elastic body, G_0 becomes

$$G_0 = \frac{h}{2v}\left(1 + \frac{\sin \nabla h}{\nabla h}\right). \tag{16}$$

As in Wang (1989a) for the plate problem, we will investigate solutions (12) and (13) in the following two sections for the cases of homogeneous boundary condition and nonhomogeneous boundary conditions, respectively.

3. THE EXACT PLANE EQUATIONS FOR HOMOGENEOUS BOUNDARY CONDITIONS

The classical plane problems consider only homogeneous boundary conditions (i.e. $p_x = q_{xs} = q_{ys} = 0$). In this case, corresponding to the three coprime factors of the determinant D_P, the general solutions of eqn (13) are the sum of the general solutions of the following governing differential equations:

$$\nabla^4\Phi_i = 0, \quad \frac{\sin(s_i\nabla z)}{s_i\nabla}\Phi_i = 0, \quad G_0\Phi_i = 0, \quad z = h/2.$$

We now discuss these three equations in detail and show how new plane theories could be established based on them.

3.1. *Biharmonic equation and biharmonic solution*

Since the solution corresponding to Φ_1 is a special case of the solution given by Φ_2 or Φ_3, we can set $\Phi_1 = 0$. Thus, eqns (12) and (13) lead to:

$$\nabla^4\Phi_2 = 0, \quad \nabla^4\Phi_3 = 0,$$

$$u = D_{21}\Phi_2 + D_{31}\Phi_3, \quad v = D_{22}\Phi_2 + D_{32}\Phi_3, \quad \psi = D_{23}\Phi_2 + D_{33}\Phi_3.$$

By replacing $(\Phi_2 h)/2$ and $(\Phi_3 h)/2$ with Φ_u and Φ_v, respectively, we can find that:

$$\nabla^4\Phi_u = 0, \quad \nabla^4\Phi_v = 0, \tag{17}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\left[\alpha_A - \alpha_B\frac{h^2}{4}\nabla^2\right]\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} + \alpha_C\nabla^2\begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix}, \tag{18}$$

$$\psi = \alpha_G\nabla^2 H; \tag{19}$$

where

$$H = \partial_x\Phi_u + \partial_y\Phi_v, \quad \alpha_G = \frac{G}{G'}, \quad \alpha_C = \alpha_A - \mu\alpha, \quad \alpha_A = \frac{1-\alpha-\mu\beta+\alpha\mu\alpha_G}{\alpha},$$

$$\alpha_B = \frac{\alpha^2[(\mu\alpha_G)^2 - 3\mu\alpha_G + 1] - (1-\mu\beta)^2}{6\alpha^2\mu},$$

and G and G' are the shear modulus in the plane of isotropy and perpendicular to it, respectively.

From eqn (1), the total displacements can be found to be

$$\begin{pmatrix} U \\ V \end{pmatrix} = \left(1 - \frac{\alpha_G z^2}{2}\nabla^2\right)\begin{pmatrix} u \\ v \end{pmatrix} + \frac{z^2}{2\alpha}\left[(\alpha\alpha_G - \beta)\begin{pmatrix} \partial_x e \\ \partial_y e \end{pmatrix} - (\alpha\alpha_G - \beta + 1)\begin{pmatrix} \partial_x \psi \\ \partial_y \psi \end{pmatrix}\right],$$

$$W = z\psi.$$

The normal stresses and shear stresses can be found to be

$$\gamma A_{13}^{-1}\sigma_{xx} = \left\{\mu\beta + \frac{z^2}{2\alpha}[\beta(1-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G(\alpha\alpha_G - \beta)\partial_{yy}]\right\}(\partial_x u + \partial_y v) - 2\alpha\mu\alpha_G\left(1 - \frac{\alpha_G z^2}{2}\nabla^2\right)\partial_y v$$
$$+ \left\{\mu(1-\alpha) + \frac{z^2}{2\alpha}[(1-\alpha^2-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G\partial_{yy}]\right\}\psi + \frac{z^4}{12\alpha}\nabla^2\left[\alpha_G(1+\alpha-\mu\beta)\partial_{yy}(\partial_x u + \partial_y v)\right.$$
$$\left. - \mu\left(\frac{\beta}{\gamma} + (\alpha\alpha_G)^2 - \beta^2\right)\partial_{yy}\right]\psi,$$

$$\gamma A_{13}^{-1}\sigma_{yy} = \left\{\mu\beta + \frac{z^2}{2\alpha}[\beta(1-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G(\alpha\alpha_G - \beta)\partial_{xx}]\right\}(\partial_x u + \partial_y v) - 2\alpha\mu\alpha_G\left(1 - \frac{\alpha_G z^2}{2}\nabla^2\right)\partial_x u$$
$$+ \left\{\mu(1-\alpha) + \frac{z^2}{2\alpha}[(1-\alpha^2-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G\partial_{xx}]\right\}\psi + \frac{z^4}{12\alpha}\nabla^2\left[\alpha_G(1+\alpha-\mu\beta)\partial_{xx}(\partial_x u + \partial_y v)\right.$$
$$\left. - \mu\left(\frac{\beta}{\gamma} + (\alpha\alpha_G)^2 - \beta^2\right)\partial_{xx}\right]\psi,$$

$$A_{13}^{-1}\sigma_{zz} = \left(1 + \frac{\beta z^2}{2(1-\alpha)}\nabla^2\right)(\partial_x u + \partial_y v) + \left(\mu + \frac{z^2}{2}\nabla^2\right)\psi,$$

$$A_{66}^{-1}\sigma_{xy} = \left(1 - \frac{\alpha_G z^2}{2}\nabla^2\right)(\partial_x v + \partial_y u) + \frac{z^2}{2\alpha}\partial_{xy}[(\alpha\alpha_G - \beta)(\partial_x u + \partial_y v) - \psi] - \frac{z^4}{24\alpha^2}\nabla^2$$
$$\times\left[\left(\frac{\beta}{\gamma} + (\alpha\alpha_G)^2 - \beta^2\right)\partial_{xy}(\partial_x u + \partial_y v) - \frac{1+\alpha-\mu\beta}{\mu}\partial_{xy}\psi\right],$$

$$A_{44}^{-1}\sigma_{xz} = -z\alpha_G\nabla^2\left(1 - \frac{\alpha_G z^2}{6}\nabla^2\right)u + z\left(\frac{\alpha\alpha_G - \beta}{\alpha} - \frac{\beta(1-\mu\beta) + \mu(\alpha\alpha_G)^2}{6\mu\alpha^2}z^2\nabla^2\right)$$
$$\times\partial_x(\partial_x u + \partial_y v) - z\left(\frac{1-\alpha}{\alpha} + \frac{1-\mu\beta-\alpha^2}{6\mu\alpha^2}z^2\nabla^2\right)\partial_x\psi,$$

$$A_{44}^{-1}\sigma_{yz} = -z\alpha_G\nabla^2\left(1 - \frac{\alpha_G z^2}{6}\nabla^2\right)v + z\left(\frac{\alpha\alpha_G - \beta}{\alpha} - \frac{\beta(1-\mu\beta) + \mu(\alpha\alpha_G)^2}{6\mu\alpha^2}z^2\nabla^2\right)$$
$$\times\partial_y(\partial_x u + \partial_y v) - z\left(\frac{1-\alpha}{\alpha} + \frac{1-\mu\beta-\alpha^2}{6\mu\alpha^2}z^2\nabla^2\right)\partial_y\psi. \quad (20)$$

Substituting (17)–(19) into the expressions for the total displacements and stresses, we obtain the following results:

$$\begin{pmatrix} U \\ V \end{pmatrix} = -\left[\alpha_A - \frac{h^2}{4}\left(\alpha_B - 2\alpha_G\left(\frac{z}{h}\right)^2\right)\nabla^2\right]\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} + \alpha_C\nabla^2\begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix}, \quad (21)$$

$$W = \alpha_G z \nabla^2 H. \tag{22}$$

and

$$\gamma A_{13}^{-1}\sigma_{xx} = -2\alpha\mu\alpha_G\left[\alpha_A - \frac{h^2}{4}\nabla^2\left(\alpha_A - 2\alpha_G\left(\frac{z}{h}\right)^2\right)\right]\partial_{xx}H + \mu\alpha\alpha_G\nabla^2[(3\alpha_A - \mu\alpha_G)H - 2\alpha_C\partial_y\Phi_v]. \tag{23}$$

$$\gamma A_{13}^{-1}\sigma_{yy} = -2\alpha\mu\alpha_G\left[\alpha_A - \frac{h^2}{4}\nabla^2\left(\alpha_A - 2\alpha_G\left(\frac{z}{h}\right)^2\right)\right]\partial_{yy}H + \mu\alpha\alpha_G\nabla^2[(3\alpha_A - \mu\alpha_G)H - 2\alpha_C\partial_x\Phi_u]. \tag{24}$$

$$A_{66}^{-1}\sigma_{xy} = \alpha_C\nabla^2(\partial_x\Phi_v + \partial_y\Phi_u) + \left[2\alpha_A + \frac{h^2}{2}\nabla^2\left(\alpha_B - 2\alpha_G\left(\frac{z}{h}\right)^2\right)\right]\partial_{xy}H. \tag{25}$$

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0. \tag{26}$$

Define the stress resultant to be

$$(N_{xx}, N_{yy}, N_{xy}) = \frac{1}{h}\int_{-h/2}^{h/2}(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})\,\mathrm{d}z, \quad (T_{zx}, T_{zy}) = \frac{1}{h}\int_{-h/2}^{h/2}(\sigma_{zz}, \sigma_{xz}, \sigma_{yz})z\,\mathrm{d}z, \tag{27}$$

then

$$\gamma A_{13}^{-1}N_{xx} = -2\alpha\mu\alpha_G\left[\alpha_A - \frac{h^2}{4}\left(\alpha_A - \frac{\alpha_G}{6}\right)\nabla^2\right]\partial_{xx}H + \mu\alpha\alpha_G\nabla^2[(3\alpha_A - \mu\alpha_G)H - 2\alpha_C\partial_y\Phi_v]. \tag{28}$$

$$\gamma A_{13}^{-1}N_{yy} = -2\alpha\mu\alpha_G\left[\alpha_A - \frac{h^2}{4}\left(\alpha_A - \frac{\alpha_G}{6}\right)\nabla^2\right]\partial_{yy}H + \mu\alpha\alpha_G\nabla^2[(3\alpha_A - \mu\alpha_G)H - 2\alpha_C\partial_x\Phi_u]. \tag{29}$$

$$A_{66}^{-1}N_{xy} = \alpha_C\nabla^2(\partial_x\Phi_v + \partial_y\Phi_u) + \left[2\alpha_A + \frac{h^2}{2}\left(\alpha_B - \frac{\alpha_G}{6}\right)\nabla^2\right]\partial_{xy}H. \tag{30}$$

$$T_{xz} = T_{yz} = 0. \tag{31}$$

Equations (17)–(19) and (21)–(31) form the *biharmonic* (plane) equation system and its solution is called the *biharmonic* (plane) solution. Equations (23)–(26) indicate that the distribution of stresses in the biharmonic solution here is the same as that of the stresses in the plane stress problem in the classical two-dimensional elasticity, i.e. both have $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. However, the plane stresses in the biharmonic solution also have the additional second-order terms (z^2 terms) which change along the thickness of planes unless $\alpha_G = 0$.

There are three major differences between the biharmonic solution and the solution of the classical plane stress problem. Firstly, the biharmonic solution is described in terms of displacement functions, and stresses in this solution are determined by differentiating the displacement functions. However, the solution of the classical plane stress problem is described in terms of stress functions, and displacements are derived by integrating the stress function. Usually, there are difficulties involved in this integrating process, especially in determining the integration constants. Secondly, there are two independent displacement functions in the biharmonic solution and only one stress function in the classical plane stress problem. However, since stresses in the biharmonic solution involve only the third- or higher-order derivatives of the displacement functions, while the stresses in the classical problem involve only the second-order derivatives of the stress function, both the biharmonic solution and the classical problem will need the same number of boundary conditions. Thirdly, the biharmonic solution satisfies all the basic equations of three-dimensional elasticity, except the three-dimensional boundary conditions. However, it is well known that the solution of the classical plane stress problem does not satisfy the three-dimensional compatible equations as well as the boundary conditions. To satisfy the three-dimensional compatible equations, the higher-order terms (z^2 terms) have to be included in the stress function (Timoshenko and Goodier, 1969). Note that by using the displacement functions, the compatible equations are automatically satisfied in the biharmonic solution.

Based on the above observation, we consider the biharmonic solution as the *first-order theory* of plane stress problems for transversely isotropic materials.

3.2. *Stretching equation and stretching solution*

It is easy to see that in this case the solution from Φ_1 is zero and the solution from Φ_3 can be reduced to that from Φ_2. Therefore, we set $\Phi_1 = \Phi_3 = 0$. Equations (12) and (13) now lead to ($\Phi = \Phi_2$)

$$\frac{\sin(s_0\nabla h/2)}{s_0\nabla}\Phi = 0,$$

$$u = D_{21}\Phi, \quad v = D_{22}\Phi, \quad \psi = D_{23}\Phi. \tag{32}$$

The transcendental differential eqn (32) can be replaced by the following infinite number of simply algebraic differential equations

$$\left[\nabla^2 - \left(\frac{2n\pi}{s_0 h}\right)^2\right]\Phi_n = 0, \quad n = 1, 2, \ldots$$

Let function Q_n be

$$Q_n = (\Sigma_{11}\Sigma_{33} - \Sigma_{13}\Sigma_{22})\partial_z\Phi_n,$$

then

$$\left[\nabla^2 - \left(\frac{2n\pi}{s_0 h}\right)^2\right]Q_n = 0, \quad n = 1, 2, \ldots \tag{33}$$

The deformation corresponding to Q_n is given by

$$u_n = \partial_y Q_n, \quad v_n = -\partial_x Q_n, \quad \psi_n = 0; \tag{34}$$

and

$$U_n = \cos\left(\frac{2n\pi z}{h}\right)\partial_y Q_n, \quad V_n = -\cos\left(\frac{2n\pi z}{h}\right)\partial_x Q_n, \quad W_n = 0. \tag{35}$$

The normal stresses and shear stresses can be found to be

$$\sigma_{xxn} = -\sigma_{yyn} = 2G\cos\left(\frac{2n\pi z}{h}\right)\partial_{xy}Q_n, \quad \sigma_{zzn} = 0, \quad \sigma_{xyn} = G\cos\left(\frac{2n\pi z}{h}\right)(\partial_{yy}Q_n - \partial_{xx}Q_n),$$

$$\sigma_{zxn} = -G'\frac{2n\pi}{h}\sin\left(\frac{2n\pi z}{h}\right)\partial_y Q_n, \quad \sigma_{zyn} = G'\frac{2n\pi}{h}\sin\left(\frac{2n\pi z}{h}\right)\partial_x Q_n. \tag{36}$$

The stress resultants are

$$N_{xxn} = N_{yyn} = N_{xyn} = 0, \quad T_{zxn} = (-1)^n G'\partial_x Q_n, \quad T_{zyn} = -(-1)^n G'\partial_y Q_n. \qquad (37)$$

The physical significance of the deformation described by (35) is very clear. Firstly, a plane parallel to the middle plane of the body slides in the same plane and hence still remains parallel, and planes at equal distances above and below the middle plane slide with the same displacements in the same directions. Therefore, overall, the body is subject to a *stretching* deformation. We call eqn (33) a *stretching equation* and the corresponding solution a *stretching solution*. Secondly, for a given n, the overall stretching deformation is composed of $2n$ layers of shear deformations. These shear deformation layers divide the thickness of the body equally into $2n$ slices,

$$\left\{ \frac{k}{n} \leqslant \frac{2z}{h} \leqslant \frac{k+1}{n} \,\middle|\, k = -n, -n+1, \ldots, -1, 0, 1, \ldots, n-1 \right\}.$$

The middle plane of the kth shear layer,

$$z = \frac{(k+\frac{1}{2})h}{2n},$$

is subject to no deformation ($U = V = W = 0$), and the two planes within the kth shear layer above and below the middle plane of the layer with equal distance slide with the same displacements but in opposite directions. Note that two adjacent shear layers are the reflection of each other with respect to their common boundary surface (Fig. 1).

For each $n = 1, 2, \ldots$, the solution (33)–(37) can be considered as an individual term in the Fourier series expansion of the stretching deformation. Similar to what we have done for plate problems (Wang, 1989a), a *refined second-order plane theory* for the deformation of transversely isotropic bodies can be established by considering only the leading term $n = 1$ in eqns (33)–(37) and combining this term with the biharmonic solution (17)–(31). A solution of the refined plane theory will satisfy all three-dimensional equilibrium equations for transversely isotropic bodies except the boundary conditions. The boundary conditions in this theory should be described in terms of the stress resultants or some combination of mid-plane displacements and rotation, instead of the stress or displacement distributions over the thickness $-h/2 \leqslant z \leqslant h/2$. Three boundary conditions at each edge of a plane can be prescribed in the refined theory. We will not discuss the problem of boundary condition here in detail.

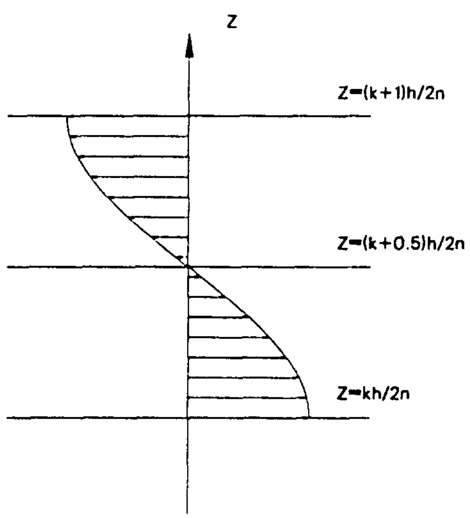

Fig. 1. The kth shear deformation layer defined by the stretching solution.

3.3. *The third equation and transcendental solution*

It is easy to see that the solution from Φ_3 can be deduced from the solution from Φ_2. Hence we set $\Phi_3 = 0$ and replace $-s_0^2SN_0\nabla^2\Phi_1$ and $-s_0^2SN_0\Phi_2$ by Φ_1 and Φ_2, then eqns (12) and (13) become:

$$G_0\Phi_1 = 0, \quad G_0\Phi_2 = 0, \tag{38}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Sigma_{33}\begin{pmatrix} \partial_x\Phi_1 \\ \partial_y\Phi_1 \end{pmatrix} + \Sigma_{13}\begin{pmatrix} \partial_x\Phi_2 \\ \partial_y\Phi_2 \end{pmatrix}, \quad \psi = (\Sigma_{22} - s_0^2SN_0)\nabla^2\Phi_1 + \Sigma_{11}\nabla^2\Phi_2. \tag{39}$$

As in the previous case, to reduce (38) to applicable differential equations, the transcendental differential operator in (38) must be replaced by an infinite number of simply algebraic operators associated with the eigenvalues of G_0 (i.e. transform G_0 into a production of infinite number of prime factors). The eigenvalues of G_0 can be found by solving the equation

$$G_0(\lambda) = 0, \tag{40}$$

which is yielded by substituting ∇^2 by λ^2 in G_0. The differential operator corresponding to an individual eigenvalue λ then becomes $\nabla^2 - \lambda^2$.

For general transversely isotropic materials, it is difficult to determine the distribution of roots of $G_0(\lambda) = 0$. However, it becomes a relatively easy task when only isotropic materials are considered. In this case, the equations for Φ_1 and Φ_2 have the form

$$\left(1 + \frac{\sin \nabla h}{\nabla h}\right)\Phi = 0, \tag{41}$$

and (40) becomes

$$\frac{\sin \lambda}{\lambda} + 1 = 0. \tag{42}$$

A similar equation, i.e. $\sin \lambda/\lambda - 1 = 0$, has been discovered for plate problems of isotropic materials by Cheng (1979), and its root distribution and other properties have been studied by Hillman and Salzer (1943).

Appendix B gives the root analysis for eqn (42). The result indicates that (42) has an (countable) infinite number of pure complex roots, and the large roots ($\lambda_n = a_n + \mathrm{j}b_n$, $\mathrm{j} = \sqrt{-1}$) have the asymptotic formulation for large n as

$$a_n \to (2n - \tfrac{1}{2})\pi - \frac{2\ln[(4n-1)\pi]}{(4n-1)\pi}, \quad b_n \to \ln[(4n-1)\pi]. \tag{43}$$

The differential equations associated with root (eigenvalue) λ_n are

$$\nabla^2\Phi_{1n} = \lambda_n^2\Phi_{1n}, \quad \nabla^2\Phi_{2n} = \lambda_n^2\Phi_{2n}, \tag{44}$$

the corresponding mid-plane displacements are

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \lambda_n^{-1}\left(\frac{\xi_n}{4\alpha}\cos\frac{\xi_n}{2} - \sin\frac{\xi_n}{2}\right)\begin{pmatrix} \partial_x\Phi_{1n} \\ \partial_y\Phi_{1n} \end{pmatrix} - \left(\mu\cos\frac{\xi_n}{2} + \frac{\xi_n}{4v}\sin\frac{\xi_n}{2}\right)\begin{pmatrix} \partial_x\Phi_{2n} \\ \partial_y\Phi_{2n} \end{pmatrix}, \quad \xi_n = \lambda_n h,$$

$$\psi_n = -\lambda_n\left(\frac{\xi_n}{4\alpha}\cos\frac{\xi_n}{2} + \sin\frac{\xi_n}{2}\right)\Phi_{1n} + \lambda_n^2\left(\cos\frac{\xi_n}{2} + \frac{\xi_n}{4v}\sin\frac{\xi_n}{2}\right)\Phi_{2n}, \tag{45}$$

and stresses and stress resultants can be found easily using (44) and (45). Note that paired conjugated roots must be used together in order to arrive at real-valued displacements and stresses. Thus only even numbered edge boundary conditions may be specified when the solution of (44) is applied alone, as has been pointed out by Cheng (1979) for plate problems.

Combining the biharmonic solution, the stretching solution and the transcendental solution just described, an infinite number of boundary conditions at the edges of planes can be satisfied. Therefore, when the boundary conditions of a general three-dimensional problem (i.e. a general three-dimensional symmetrical deformation problem) can be described by an infinite number of two-dimensional terms, its solution can be approximated by the combination of those basic solutions. Some relevant discussion about this problem can be found in Cheng (1977).

4. APPROXIMATE PLANE EQUATIONS FOR NONHOMOGENEOUS BOUNDARY CONDITIONS

For nonhomogeneous boundary conditions, we first discuss the most general loading condition at the two surfaces $z = \pm h/2$, and then consider two special cases of boundary conditions, that is, normal surface loads only and shear surface loads only. As in the previous section, we only present the approximate governing equations and the approximate expressions for displacements and stresses. Issues related with the corresponding boundary conditions at edges will not be investigated here.

4.1. *General boundary condition at the two surfaces* $z = \pm h/2$

In this case all the three load terms p_e, q_{xs} and q_{ys} are applied on the two surfaces. The governing equations for mid-plane displacements and rotation can be obtained from (10) by approximating all the differential operators up to second-order terms with respect to the thickness h. The result is the following:

$$\left(1+\frac{\beta h^2}{8(1-\alpha)}\nabla^2\right)(\partial_x u+\partial_y v)+\left(\mu+\frac{h^2}{8}\nabla^2\right)\psi = \frac{p_s}{A_{13}},$$

$$-\alpha_G\left(1-\frac{\alpha_G h^2}{24}\nabla^2\right)\nabla^2 u+\left[\frac{\alpha\alpha_G-\beta}{\alpha}-\frac{\beta(1-\mu\beta)+\mu\alpha^2\alpha_G^2}{24\alpha^2\mu}h^2\nabla^2\right]\partial_x(\partial_x u+\partial_y v)$$

$$-\left(\frac{1-\alpha}{\alpha}+\frac{1-\alpha^2-\mu\beta}{24\alpha^2\mu}h^2\nabla^2\right)\partial_x\psi = \frac{2q_{xs}}{A_{44}h},$$

$$\left[\frac{\alpha\alpha_G-\beta}{\alpha}-\frac{\beta(1-\mu\beta)+\mu\alpha^2\alpha_G^2}{24\alpha^2\mu}h^2\nabla^2\right]\partial_y(\partial_x u+\partial_y v)-\alpha_G\left(1-\frac{\alpha_G h^2}{24}\nabla^2\right)\nabla^2 v-$$

$$\left(\frac{1-\alpha}{\alpha}+\frac{1-\alpha^2-\mu\beta}{24\alpha^2\mu}h^2\nabla^2\right)\partial_y\psi = \frac{2q_{ys}}{A_{44}h}. \quad (46)$$

The approximate expressions for three-dimensional displacements are obtained from (1) by considering operators up to second-order terms with respect to z. We get

$$\begin{pmatrix}U\\V\end{pmatrix} = \left(1-\frac{\alpha_G z^2}{2}\nabla^2\right)\begin{pmatrix}u\\v\end{pmatrix}+\frac{z^2}{2\alpha}\left[(\alpha\alpha_G-\beta)\begin{pmatrix}\partial_x e\\\partial_y e\end{pmatrix}-(\alpha\alpha_G-\beta+1)\begin{pmatrix}\partial_x\psi\\\partial_y\psi\end{pmatrix}\right],$$

$$W = z\psi. \quad (47)$$

Similarly, from (3), (4) and (6), we can show that

$$\gamma A_{13}^{-1}\sigma_{xx} = \left\{\mu\beta + \frac{z^2}{2\alpha}[\beta(1-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G(\alpha\alpha_G-\beta)\partial_{yy}]\right\}(\partial_x u+\partial_y v)$$

$$-2\alpha\mu\alpha_G\left(1-\frac{\alpha_G z^2}{2}\nabla^2\right)\partial_y v + \left\{\mu(1-\alpha) + \frac{z^2}{2\alpha}[(1-\alpha^2-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G\partial_{yy}]\right\}\psi,$$

$$\gamma A_{13}^{-1}\sigma_{yy} = \left\{\mu\beta + \frac{z^2}{2\alpha}[\beta(1-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G(\alpha\alpha_G-\beta)\partial_{xx}]\right\}(\partial_x u+\partial_y v) -$$

$$2\alpha\mu\alpha_G\left(1-\frac{\alpha_G z^2}{2}\nabla^2\right)\partial_x u + \left\{\mu(1-\alpha) + \frac{z^2}{2\alpha}[(1-\alpha^2-\mu\beta)\nabla^2 - 2\alpha\mu\alpha_G\partial_{xx}]\right\}\psi,$$

$$\frac{\sigma_{xy}}{G} = \left(1-\frac{\alpha_G z^2}{2}\nabla^2\right)(\partial_x v+\partial_y u) + \frac{z^2}{2\alpha}\partial_{xy}[(\alpha\alpha_G-\beta)(\partial_x u+\partial_y v)-\psi]. \quad (48)$$

Combining (5), (7) and (8) with (46), we have:

$$\frac{\sigma_{zz}}{A_{13}} = \frac{p_s}{A_{13}} - \frac{1}{2}\left(\frac{h^2}{4}-z^2\right)\nabla^2\left[\frac{\beta}{1-\alpha}(\partial_x u+\partial_y v)-\psi\right],$$

$$\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} = \frac{2z}{h}\begin{pmatrix}q_{xs}\\ q_{ys}\end{pmatrix} - \frac{G'}{6}z\left(\frac{h^2}{4}-z^2\right)\nabla^2\left[\alpha_G^2\nabla^2\begin{pmatrix}u\\ v\end{pmatrix}\right.$$

$$\left. -\frac{\beta(1-\mu\beta)+\mu(\alpha\alpha_G)^2}{\mu\alpha^2}\begin{pmatrix}\partial_x(\partial_x u+\partial_y v)\\ \partial_y(\partial_x u+\partial_y v)\end{pmatrix} + \frac{1-\mu\beta-\alpha^2}{\mu\alpha^2}\begin{pmatrix}\partial_x\psi\\ \partial_y\psi\end{pmatrix}\right]. \quad (49)$$

Equations (49) show that all the boundary conditions at the two surfaces are completely satisfied.

Another approach to develop approximate equations in this case is to use eqns (12) and (13). However, we will not discuss this approach since we can show later that such a result is simply the superposition of results from the cases in 4.2 and 4.3 below.

4.2. *Two surfaces subject to normal loads only*

A typical case for such loading is a dam with constant width h pressured by water. Since $X_2 = X_3 = 0$ in this case, we can set $\Phi_2 = \Phi_3 = 0$ in (13). Let $\Phi = -s_0^2\nabla^2 SN_0\Phi_1$; eqns (12) and (13) then can be rewritten as

$$\nabla^2 G_0\Phi = -\frac{p_s}{A_{13}}, \quad u = \Sigma_{13}\partial_x\Phi, \quad v = \Sigma_{13}\partial_y\Phi, \quad \psi = \nabla^2(\Sigma_{22}-\alpha_G SN_0)\Phi.$$

Expanding G_0 up to the fourth order with respect to z, we get

$$G_0 = -z\alpha_G(1+\alpha_H z^2\nabla^2), \quad z = h/2$$

where

$$\alpha_H = \frac{1+\alpha-\mu\beta}{6\alpha\mu}.$$

Therefore, the approximate governing equation for Φ is (replacing $h\Phi/2$ by Φ)

$$\nabla^2\Phi = \left(1 - \frac{\alpha_H h^2}{4}\nabla^2\right)\frac{p_s}{\alpha_C A_{13}}, \tag{50}$$

and the approximate mid-plane displacements and rotation are

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1-\alpha}{\alpha}\begin{pmatrix} \partial_x\Phi \\ \partial_y\Phi \end{pmatrix} + \frac{1-\alpha^2-\mu\beta}{24\alpha^2\mu}h^2\frac{1}{\alpha_C A_{13}}\begin{pmatrix} \partial_x p_s \\ \partial_y p_s \end{pmatrix}, \quad \psi = \frac{\beta}{\alpha}\left[1 - \frac{1}{24\mu}h^2\nabla^2\right]\frac{p_s}{\alpha_C A_{13}}. \tag{51}$$

From (1), the approximate three-dimensional displacements are

$$\begin{pmatrix} U \\ V \end{pmatrix} = \frac{1-\alpha}{\alpha}\begin{pmatrix} \partial_x\Phi \\ \partial_y\Phi \end{pmatrix} + \frac{1}{2\alpha}\left(\frac{1-\alpha^2-\mu\beta}{12\alpha\mu}h^2 + \beta z^2\right)\frac{1}{\alpha_C A_{13}}\begin{pmatrix} \partial_x p_s \\ \partial_y p_s \end{pmatrix}. \tag{52}$$

Combining (3)–(8) with (50), we find the approximate stresses as

$$\begin{aligned}
\gamma\frac{\sigma_{xx}}{A_{13}} &= 2(1-\alpha)\mu\alpha_G\partial_{yy}\Phi + \left\{\frac{\beta}{2}\left(\frac{h^2}{12} - z^2\right)\nabla^2 - \frac{\alpha_G}{\alpha\alpha_C}\left[(1-\alpha^2-\mu\beta)\frac{h^2}{12} - \mu\beta(2-\alpha)z^2\right]\partial_{yy}\right\}\frac{p_s}{A_{13}}, \\
\gamma\frac{\sigma_{yy}}{A_{13}} &= 2(1-\alpha)\mu\alpha_G\partial_{xx}\Phi + \left\{\frac{\beta}{2}\left(\frac{h^2}{12} - z^2\right)\nabla^2 - \frac{\alpha_G}{\alpha\alpha_C}\left[(1-\alpha^2-\mu\beta)\frac{h^2}{12} - \mu\beta(2-\alpha)z^2\right]\partial_{xx}\right\}\frac{p_s}{A_{13}}, \\
\frac{\sigma_{xy}}{G} &= \frac{2(1-\alpha)}{\alpha}\partial_{xy}\Phi + \frac{1}{2\alpha\alpha_C}\left[\frac{1-\alpha^2-\mu\beta}{6\mu\alpha}h^2 - (\alpha_G(1-\alpha)-\beta)z^2\right]\partial_{xy}\frac{p_s}{A_{13}} \\
\sigma_{zz} &= \left[1 - \frac{\beta}{\alpha\alpha_C}\left(\frac{h^2}{4} - z^2\right)\nabla^2\right]p_s, \\
\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} &= \frac{(2-\alpha)(1-\mu\beta)-\alpha^2}{6\alpha\gamma(1-\alpha-\mu\beta)}\beta z\left(\frac{h^2}{4} - z^2\right)\nabla^2\begin{pmatrix} \partial_x p_s \\ \partial_y p_s \end{pmatrix}.
\end{aligned} \tag{53}$$

Again, the boundary conditions at the two surfaces are satisfied completely. The last equation in (53) indicates that stresses σ_{zz}, σ_{zx} and σ_{zy} are caused totally by the normal surface load p_s.

4.3. *Two surfaces subject to shear loads only*

Since $X_1 = 0$ in this case, we can set $\Phi_1 = 0$ in (13) and rewrite Φ_2 and Φ_3 as Φ_u and Φ_v. After some quite tedious calculation, the approximate governing equations for Φ_u and Φ_v can be found to be

$$\nabla^4\Phi_u + \left(1 - \frac{\alpha_P h^2}{4}\nabla^2\right)\frac{q_{xs}}{C} = 0, \quad \nabla^4\Phi_v + \left(1 - \frac{\alpha_P h^2}{4}\nabla^2\right)\frac{q_{ys}}{C} = 0, \tag{54}$$

where

$$\alpha_P = \alpha_H - \frac{\alpha_G}{6} \quad \text{and} \quad C = \frac{\alpha_C G h}{2}.$$

The approximate mid-plane displacements and rotation in terms of Φ_u and Φ_v are

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\left[\alpha_A - \alpha_B \frac{h^2}{4}\nabla^2\right]\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} + \alpha_C \nabla^2 \begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix} - \alpha_H \alpha_C \frac{h^2}{4C}\begin{pmatrix} q_{xs} \\ q_{ys} \end{pmatrix},$$

$$\frac{\psi}{\alpha_G} = \nabla^2 H + \frac{\alpha_G(1-\alpha) - 3\beta}{6(1-\alpha)} \frac{h^2}{4C} p; \tag{55}$$

where $p = \partial_x q_{xs} + \partial_y q_{ys}$.

The three-dimensional displacements are approximated by

$$\begin{pmatrix} U \\ V \end{pmatrix} = \alpha_C \nabla^2 \begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix} - \tfrac{1}{4}\{4\alpha_A - \alpha_B h^2 \nabla^2 - 2\mu[\alpha\beta\mu - \alpha_G(\alpha + \mu\beta)]z^2\nabla^2\}\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix}$$
$$- \frac{\alpha_C}{4C}(\alpha_C h^2 - 2\alpha_G z^2)\begin{pmatrix} q_{xs} \\ q_{ys} \end{pmatrix},$$

$$W = z\psi; \tag{56}$$

and finally, the approximate stresses are described by

$$\gamma \frac{\sigma_{xx}}{A_{13}} = -2\alpha\mu\alpha_G\alpha_C \nabla^2 \partial_y \Phi_v + \mu[(\alpha_G(1-\alpha) - \alpha\beta\mu)\nabla^2 + 2\alpha\alpha_A\alpha_G\partial_{yy}]H - \frac{\mu\alpha_G}{2}\{\alpha\alpha_B h^2$$
$$+ 2[\alpha\beta\mu - \alpha_G(1 + \alpha\alpha_A - \mu\alpha^2)]z^2\}\nabla^2\partial_{yy}H + \frac{\mu\alpha\alpha_G}{Gh}(\alpha_H h^2 - 2\alpha_G z^2)\partial_y q_{ys} + \frac{1}{4\alpha C}\left[\frac{\alpha\alpha_C\alpha_G^2}{6}h^2\right.$$
$$\left. + 2((\alpha\beta\mu - \alpha_G)(1 - \mu\beta) + \alpha_G\alpha^2)z^2\right]p,$$

$$\gamma \frac{\sigma_{yy}}{A_{13}} = -2\alpha\mu\alpha_G\alpha_C \nabla^2 \partial_x \Phi_u + \mu[(\alpha_G(1-\alpha) - \alpha\beta\mu)\nabla^2 + 2\alpha\alpha_A\alpha_G\partial_{xx}]H - \frac{\mu\alpha_G}{2}\{\alpha\alpha_B h^2$$
$$+ 2[\alpha\beta\mu - \alpha_G(1 + \alpha\alpha_A - \mu\alpha^2)]z^2\}\nabla^2\partial_{xx}H + \frac{\mu\alpha\alpha_G}{Gh}(\alpha_H h^2 - 2\alpha_G z^2)\partial_x q_{xs} + \frac{1}{4\alpha C}\left[\frac{\alpha\alpha_C\alpha_G^2}{6}h^2\right.$$
$$\left. + 2((\alpha\beta\mu - \alpha_G)(1 - \mu\beta) + \alpha_G\alpha^2)z^2\right]p,$$

$$\frac{\sigma_{xy}}{G} = 2\alpha_A\partial_{xy}H + \alpha_C\nabla^2(\partial_x\Phi_v + \partial_y\Phi_u) + \frac{1}{2\alpha}\{\alpha\alpha_B h^2 + [\alpha\beta\mu - \alpha_G(1 - 2\alpha\alpha_A + \mu\alpha^2)]z^2\}$$
$$\times \nabla^2\partial_{xy}H - \frac{\alpha_C}{4C}(\alpha_H h^2 - 2\alpha_G z^2)p,$$

$$\sigma_{zz} = \frac{\alpha_G(1-\alpha) + \alpha\beta\mu}{\alpha\alpha_C\alpha_G} h\left[\frac{1}{4} - \left(\frac{z}{h}\right)^2\right]p,$$

$$\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \frac{2z}{h}\left\{\left[1 - \frac{\alpha_G}{6}\left(\frac{h^2}{4} - z^2\right)\nabla^2\right]\begin{pmatrix} q_{xs} \\ q_{ys} \end{pmatrix} - \frac{(\alpha_G + \alpha\beta\mu)(1 - \mu\beta) - \alpha^2\alpha_G(1 + \alpha_G\alpha_C\mu)}{6\mu\alpha^2\alpha_G\alpha_C}\right.$$
$$\left. \times \left(\frac{h^2}{4} - z^2\right)\begin{pmatrix} \partial_x p \\ \partial_y p \end{pmatrix}\right\}. \tag{57}$$

Once more, the boundary conditions at the two surfaces are satisfied completely and the stresses σ_{zz}, σ_{zx} and σ_{zy} are caused totally by the shear surface loads.

The stress resultants in the above three cases can be found easily.

To verify the statement we made at the end of case 4.1, substituting mid-plane displacements and rotation of (51) and (55) into (47)–(49) and neglecting the higher-order terms, we find that the result obtained in this way is indeed the superposition of (52)–(53) and (56)–(57). However, for the general boundary condition (case 4.1), there does exist a higher-order term difference between the solution by solving (46)–(49) and that by superposing the results of the cases in 4.2 and 4.3. Obviously, it is more convenient and preferred to solve the general surface boundary condition by dealing with the cases in 4.2 and 4.3 separately.

5. CONCLUSION AND FUTURE RESEARCH

Without using any *ad hoc* assumptions, we have deduced various two-dimensional exact and approximate equations for the plane problems systematically and directly from the three-dimensional theory of transversely isotropic bodies. These equations are very useful in developing new refined theories for the plane problems. Note the plane problem defined here is a generalization of the plane stress problem studied in classical two-dimensional elasticity for isotropic materials, where only homogeneous boundary conditions at the two surfaces are considered. In the case of homogeneous boundary conditions, the equations obtained for the plane problems in this paper are exact ones in the sense that a solution of them will satisfy all the balance equations of the three-dimensional theory. Especially, the distribution of stresses described by the biharmonic solution is the same as that of stresses in the classical plane stress problem (i.e. both have $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$). In the case of nonhomogeneous boundary conditions, the approximate equations are accurate up to the second-order terms with respect to plane thickness. Without considering the higher-order terms, stresses σ_{zz}, σ_{xz} and σ_{yz} described by the approximate equations are totally due to the external surface loads. Those results obtained for stresses σ_{zz}, σ_{xz} and σ_{yz} indicate the correctness of the stress assumption ($\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$) in the classical plane stress problem.

In the plate problems the equations of the refined theories have the same form as that of other well-known classical plate theories. This property is very helpful to show the validity of the new theories. For the plane problems, however, no direct comparison on the form of equations can be made since the unknown function in the classical plane problem is the stress function, while the unknowns in the refined theories would be displacement functions. Therefore, further research is required to verify the new plane equations. One of the most important issues to be investigated is the specification of boundary conditions at the edges of planes. These boundary conditions have to be specified in terms of the stress resultants (N_{xx}, N_{yy}, N_{xy} and T_{xz}, T_{yz}) or some combinations of mid-plane displacements and rotation (u, v and ψ). Even though the stress resultants T_{xz} and T_{yz} defined here have not been widely (perhaps never) used before, the correspondence between (N_{xx}, N_{yy}, N_{xy}, T_{xz}, T_{yz}) and (M_{xx}, M_{yy}, M_{xy}, V_{xz}, V_{yz}) in plate theory indicates they are important physical quantities and more understanding about them is needed.

Another aspect of future research is to apply the established equations to some specific problems, especially the fracture problem and stress concentration problem, and compare the results obtained with those by both the classical two-dimensional approximate theory and the three-dimensional exact theory.

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APPENDIX A: CONSTITUTIVE EQUATIONS, MATERIAL COEFFICIENTS AND DIFFERENTIAL OPERATORS

A.1. *Constitutive equations*

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{12} & A_{11} & A_{13}\\ A_{13} & A_{13} & A_{33}\end{bmatrix}\begin{pmatrix}\partial_x U\\ \partial_y V\\ \partial_z W\end{pmatrix}$$

$$\sigma_{zx} = A_{44}(\partial_x W+\partial_z U),\quad \sigma_{zy} = A_{44}(\partial_y W+\partial_z V),\quad \sigma_{xy} = A_{66}(\partial_y U+\partial_x V),\quad \text{where}\quad A_{66} = (A_{11}-A_{12})/2.$$

A.2. *Some coefficients*

$$\alpha = \frac{A_{44}}{A_{13}+A_{44}},\quad \beta = \frac{A_{11}}{A_{13}+A_{44}},\quad \gamma = \frac{A_{33}}{A_{13}+A_{44}},\quad \mu = \frac{A_{33}}{A_{13}},\quad \alpha\mu = \mu-\gamma.$$

$$s_0^2 = \frac{A_{66}}{A_{44}},\quad s_{12}^2 = B\pm\sqrt{B^2-\frac{\beta}{\gamma}},\quad B = \frac{\alpha^2+\beta\gamma-1}{2\alpha\gamma}.$$

For isotropic materials: $\alpha = 1-2\nu,\ \beta = \gamma = 2(1-\nu),\quad \mu = (1-\nu)/\nu.$

A.3. *Basic differential operators*

$$SN_i = \frac{\sin(s_i\nabla z)}{s_i\nabla},\quad CS_i = \cos(s_i\nabla z),\quad SS_i = \frac{z-SN_i}{s_i^2\nabla^2},\quad CC_i = \frac{1-CS_i}{s_i^2\nabla^2},\quad i = 0,1,2.$$

$$\Omega^i_{ss} = \frac{s_1^{2i}SS_1-s_2^{2i}SS_2}{s_1^2-s_2^2},\quad \Omega^i_{cc} = \frac{s_1^{2i}CC_1-s_2^{2i}CC_2}{s_1^2-s_2^2},\quad i = 0,1,2,3,$$

when $s_0\to 1,\ s_1\to 1,\ s_2\to 1$:

$$\Omega^i_{ss}\to(i-1)\frac{z-\dfrac{\sin\nabla z}{\nabla}}{\nabla^2}-\frac{1}{2}\frac{z\cos\nabla z-\dfrac{\sin\nabla z}{\nabla}}{\nabla^2},\quad \Omega^i_{cc}\to(i-1)\frac{1-\cos\nabla z}{\nabla^2}+\frac{z}{2}\frac{\sin\nabla z}{\nabla}.$$

A.4. *Some differential relationships*

$$\partial_z SN_i = CS_i,\quad \partial_z CS_i = -s_i^2\nabla^2 SN_i,\quad \partial_z SS_i = CC_i,\quad \partial_z CC_i = SN_i,\quad i = 0,1,2.$$

$$\partial_z\Omega^i_{ss} = \Omega^i_{cc},\quad \partial_z\Omega^i_{cc} = z\frac{s_1^{2i}-s_2^{2i}}{s_1^2-s_2^2}-\nabla^2\Omega^{i+1}_{ss},\quad i = 0,1,2,3,$$

$$\partial_z L_\psi = 1-\nabla^2\left(\Omega^2_{cc}-\frac{\beta}{\alpha}\Omega^1_{cc}\right),\quad \partial_z L_w = s_0^2 SN_0-\frac{\beta}{\alpha}z+\nabla^2\left(\frac{\beta}{\alpha}\Omega^2_{ss}-\frac{\beta}{\gamma}\Omega^1_{ss}\right),\quad \partial_z L_{1\psi} = \frac{z}{\alpha}-\frac{1}{\alpha}\nabla^2\Omega^2_{ss},$$

$$\partial_z L_{2w} = \frac{\beta}{\alpha\gamma}\nabla^2[\alpha\gamma\Omega^2_{cc}-(\alpha^2+\beta\gamma)\Omega^1_{cc}+\alpha\beta\Omega^0_{cc}].$$

A.5. *Cofactors D_{ij}*

$$D_{11} = -s_0^2 SN_0\nabla^2\Sigma_{33}\partial_x,\quad D_{12} = -s_0^2 SN_0\nabla^2\Sigma_{33}\partial_y,\quad D_{13} = s_0^2 SN_0\nabla^4(s_0^2 SN_0-\Sigma_{22}),$$

$$D_{21} = s_0^2 SN_0\nabla^2\Sigma_{13}+(\Sigma_{11}\Sigma_{33}-\Sigma_{13}\Sigma_{22})\partial_{yy},\quad D_{22} = -(\Sigma_{11}\Sigma_{33}-\Sigma_{13}\Sigma_{22})\partial_{xy},\quad D_{23} = s_0^2 SN_0\nabla^2\Sigma_{11}\partial_x,$$

$$D_{31} = -(\Sigma_{11}\Sigma_{33}-\Sigma_{13}\Sigma_{22})\partial_{xy},\quad D_{32} = s_0^2 SN_0\nabla^2\Sigma_{13}+(\Sigma_{11}\Sigma_{33}-\Sigma_{13}\Sigma_{22})\partial_{xx},\quad D_{33} = s_0^2 SN_0\nabla^2\Sigma_{11}\partial_y.$$

APPENDIX B: ROOTS OF $\sin\lambda/\lambda+1 = 0$

First, it is easy to show that equation

$$\sin\lambda/\lambda+1 = 0 \tag{B1}$$

has no pure real or imaginary roots.

Let $x = a+jb$ be a complex root, a and b be real numbers. Then $\pm a\pm jb$ are roots too. Therefore we only need to consider the region $\{a > 0,\ b > 0\}$. Substituting x into (B1), we get

$$\begin{cases}\sin(a)\cosh(b)+a = 0\\ \cos(a)\sinh(b)+b = 0.\end{cases} \tag{B2}$$

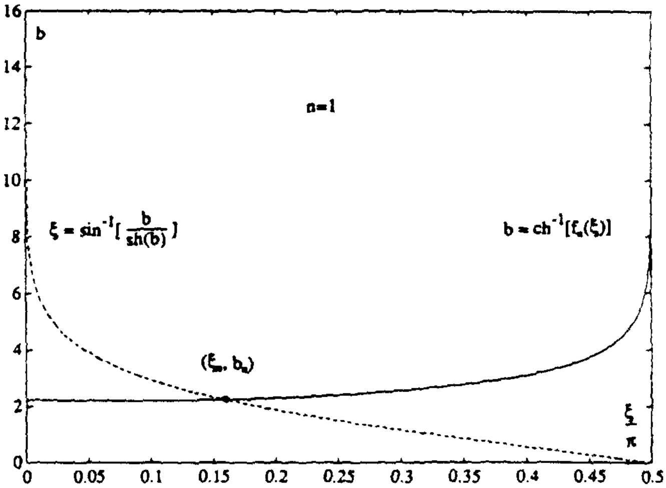

Fig. B1(a). Determination of root for $n = 1$.

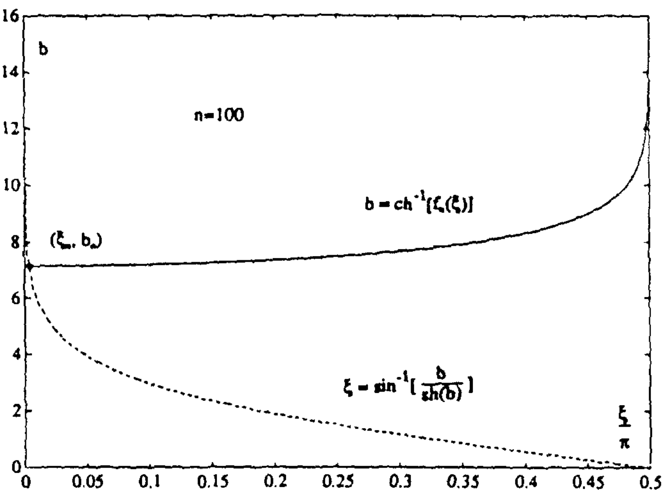

Fig. B1(b). Determination of root for $n = 100$.

B.1. *Countability of roots*

Clearly, eqn (B2) may have real roots only when $\sin(a) < 0$ and $\cos(a) < 0$. Hence a must satisfy

$$(2n-1)\pi < a < (2n-\tfrac{1}{2})\pi, \quad n = 1, 2, \ldots,$$

therefore

$$a = (2n-\tfrac{1}{2})\pi - \xi, \quad 0 < \xi < \frac{\pi}{2}. \tag{B3}$$

Substitute a into (B2),

$$\begin{cases} \cos(\xi)\cosh(b) = (2n-\tfrac{1}{2})\pi - \xi \\ \sin(\xi)\sinh(b) = b. \end{cases} \tag{B4}$$

Define

$$f_n(\xi) = \frac{(2n-\tfrac{1}{2})\pi - \xi}{\cos(\xi)} \quad \text{and} \quad \alpha_n(\xi) = \sin(\xi) f_n(\xi),$$

then from (B4)

$$\tanh(b) = \frac{b}{\alpha_n(\xi)}$$

which may have positive real roots only when $\alpha_n(\xi) > 1$.

Let ξ_n^0 be the root of the equation

$$\xi = (2n - \tfrac{1}{2})\pi - \frac{\cos(\xi)}{\sin(\xi)}, \quad 0 < \xi < \frac{\pi}{2}$$

then ξ_n^0 is uniquely defined for any $n \geq 1$. It can be shown easily that $f_n(\xi)$ is a decreasing function of ξ when $0 < \xi \leq \xi_n^0$, and an increasing function when $\xi_n^0 < \xi < (\pi/2)$; and that $\alpha_n(\xi) \leq 1$ when $0 < \xi \leq \xi_n^0$, $\alpha_n(\xi) > 1$ when $\xi_n^0 < \xi < (\pi/2)$. Therefore, for any n in (B3)

$$\xi_n^0 < \xi < \frac{\pi}{2}. \tag{B5}$$

However, from (B4),

$$\xi = \sin^{-1}\left[\frac{h}{\sinh(h)}\right] \tag{B6}$$

$$h = \cosh^{-1}[f_n(\xi)]. \tag{B7}$$

(B6) indicates that ξ is a decreasing function of h for $0 \leq h \leq \infty$, and (B7) indicates that h is an increasing function of ξ for $\xi_n^0 < \xi < (\pi/2)$. Also, from (B6) and (B7)

$$\xi \to \frac{\pi}{2} \quad \text{when} \quad h \to 0 \quad \text{and} \quad h \to \infty \quad \text{when} \quad \xi \to \frac{\pi}{2}.$$

Therefore, for any given n there exists one and only one ξ which satisfies (B3) and one and only one h which satisfies (B2). This proves that (B1) has countable infinite number of roots. For a given n, we will write the corresponding a, h and ξ as a_n, h_n and ξ_n. Figure B1(a) and (b) depict eqns (B6) and (B7) for $n = 1$ and $n = 100$, respectively.

B.2. *Asymptotic distribution of roots*

It is not difficult to give a rigorous analysis for the asymptotic distribution of roots, but here we will use a rather simple method to determine the asymptotic formulation for large roots.

From (B3), $a_n \to \infty$ when $n \to \infty$, therefore by (B2) and (B6), $h_n \to \infty$ and $\xi_n \to 0$. Since

$$\sin(\xi_n) \to \xi_n \quad \text{when} \quad \xi_n \to 0 \quad \text{and} \quad \cosh(h_n) \to \frac{e^{h_n}}{2}, \quad \sinh(h_n) \to \frac{e^{h_n}}{2} \quad \text{when} \quad h_n \to \infty$$

it follows from (B3),

$$h_n \to \ln[(4n-1)\pi] \quad \text{and} \quad \xi_n \to \frac{2\ln[(4n-1)\pi]}{(4n-1)\pi} \quad \text{when} \quad n \to \infty \tag{B8}$$

which give the asymptotic formulation for large roots.